Graphs with Large Overlap in Their Spanning Trees

Pablo Blanco*

Abstract

The overlap of a spanning tree of a graph is one way to measure how much its fundamental cycles intersect. In this paper, we show that there is a family of graphs with bounded treewidth and bounded degree but no spanning tree of small overlap. This paper also discusses a connection to a recently answered question by Zdeněk Dvořák.

1 Introduction

The main theorem of this paper, presented below, tells us that the overlap parameter is not bounded by any function of treewidth for general graphs. We define both parameters later in this section.

Theorem 1. *For every integer $k \geq 2$, there is a connected graph G with maximum degree 3 such that $\mathbf{tw}(G) = 2$ and $\mathbf{overlap}(G) \geq k$.*

A *tree decomposition* of a connected graph G is a pair $(T, \mathcal{B})$ such that T is a tree and $\mathcal{B}$ is a function which maps each $x \in V(T)$ to a set $\mathcal{B}(x) \subseteq V(G)$, with the following properties:

- The collection $\{\mathcal{B}(x) : x \in V(T)\}$ covers $V(G)$. That is, we have $\bigcup_{x \in V(T)} \mathcal{B}(x) = V(G)$.
- For every edge $uw \in E(G)$, there is some $x \in V(T)$ such that $u, w \in \mathcal{B}(x)$.
- For every $u \in V(G)$, its preimage $\mathcal{B}^{-1}(u) \subseteq V(T)$ induces a connected subgraph of T.

The *width* of a tree decomposition $(T, \mathcal{B})$ is the maximum of $|\mathcal{B}(x)| - 1$ taken over $x \in V(T)$. The *treewidth* of a graph G, denoted $\mathbf{tw}(G)$, is the minimum width over its tree decompositions.

Given a spanning tree T of G, we denote the *fundamental cycle* of an edge $e \in E(G) \setminus E(T)$ with respect to T by ${C_T}^e$; we sometimes write ${P_T}^e := {C_T}^e - e$. We define the *overlap* of a spanning tree T with respect to G, denoted by $\mathbf{overlap}(T, G)$, as the maximum size of a set $S \subseteq E(G) \setminus E(T)$ which satisfies $\left|\bigcap_{e \in S} E({P_T}^e)\right| \geq |S|$. Whenever $|S| \geq 2$, it is equivalent to consider $\bigcap_{e \in S} E({C_T}^e)$ and we will use this fact throughout the paper.

For a graph G, we define its *overlap* to be the minimum $\mathbf{overlap}(T, G)$ across all of its spanning trees and denote it by $\mathbf{overlap}(G)$. For convenience of notation, we will often write $\mathbf{overlap}(T)$ instead of $\mathbf{overlap}(T, G)$ whenever G is clear from context.

Informally, the overlap tells us if a graph has a spanning tree with sets of fundamental cycles that do not intersect much. In particular, if the overlap of a graph is equal to $b \geq 2$ and T is a spanning tree with overlap b, then we can deduce two facts:

*Department of Mathematics, Princeton University, Princeton, USA.

- If $\mathcal{C}$ is a collection of fundamental cycles from T with $|\mathcal{C}| \geq b$, then $|\bigcap_{C \in \mathcal{C}} E(C)| \leq b$.
- Any spanning tree T' of the graph has a collection of fundamental cycles $\mathcal{C}_{T'}$ of size b such that $|\bigcap_{C \in \mathcal{C}} E(C)| \geq b$.

Loopless graphs with overlap 0 are exactly trees. One might expect that if a graph "looks" like a tree according to the treewidth, then it also has a nice spanning tree in terms of overlap; however, Theorem 1 rejects this notion strongly. We give an outline of further sections below.

In Section 2, we give further background on treewidth and related parameters; such as the treedepth, which can bound the longest path of a graph. In Section 3, we briefly discuss a weaker notion of overlap and Corollary 7 which the author, alongside [2], used to prove the theorem below (and resolved a problem posed by Zdeněk Dvořák in [4]):

Theorem 2. [2] *For every positive integer k, there is a connected graph G of treewidth 2 such that if $(T, \mathcal{B})$ is a tree decomposition of G and T is a minor of G, then $(T, \mathcal{B})$ has width at least k.*

Robert Hickingbotham independently proved Corollary 7 in [5, Theorem 7.5.1] with a different construction. Section 4 provides most of the set-up for the proof of Theorem 1, while Section 5 introduces the most important ideas and proves Theorem 1. Lastly, we discuss future directions in Section 6 while proving further properties of the overlap and providing a conjecture regarding a minor-closed class with bounded overlap.

2 Background on Treewidth Boundedness

In this section, we discuss some facts relating to treewidth, related parameters, and their connection to overlap. Any graphs are assumed to be simple unless stated otherwise. We will often use the notation below.

Notation. We denote $[k] := \{1, ..., k\}$.

The concept of treewidth was first introduced in 1972 under a different name by Umberto Bertelè and Francesco Brioschi [1]. In their book, they define a parameter they call the (scalar) dimension of a graph. Consider an ordering of $V(G)$ given by $\Omega = (v_1, v_2, \dots, v_n)$; the graph G_i^Ω with $i \in \{0, 1, \dots, n\}$ is obtained from G_{i-1}^Ω by making every vertex in $N(v_i) \cap \{v_j : j > i\}$ adjacent and then deleting v_i with $G_0^\Omega := G$. The *scalar dimension* of G is

$$\min_{\Omega} \left\{ \max_{i \in \{1, \dots, n\}} \deg_{G_{i-1}^\Omega}(v_i) \right\}$$

over all vertex orderings Ω of $V(G)$. We will prove that the *scalar dimension* of G is the same as its treewidth. First, we begin with a lemma of tree decompositions.

Lemma 3. *Let $(T, \mathcal{B})$ be a tree decomposition of G. If K is a clique of G, then there is some $x \in V(T)$ for which $K \subseteq \mathcal{B}(x)$.*

Proof. We may assume $|K| \geq 2$. Suppose the claim does not hold for a contradiction. Let T' be a minimal subtree of T such that $K \subseteq \bigcup_{u \in V(T)} \mathcal{B}(u)$. By assumption, $|V(T')| \geq 2$ so we can find two distinct leaves $u, u' \in V(T')$. By the minimality of T', there are distinct vertices $v, v' \in K$ such that $\mathcal{B}^{-1}(v) \cap V(T') = \{u\}$ and $\mathcal{B}^{-1}(v') \cap V(T') = \{u'\}$. However, because $vv' \in E(G)$, we can find a vertex of $V(T')$ in $\mathcal{B}^{-1}(\{v, v'\})$, but this contradicts the minimality of T'! ☐

The following lemma shows the equivalence between the scalar dimension of a graph and its treewidth.

Lemma 4. *A graph G on n vertices has scalar dimension t if and only if* $\mathbf{tw}(G) = t$.

Proof. We will say that a vertex ordering Ω of G has *dimension m* if

$$\max_{i \in \{1,\dots,n\}} \deg_{G_{i-1}^{\Omega}}(v_i).$$

For this proof, we will prove the stronger claim that a graph G has a vertex ordering of dimension t if and only if it has a tree decomposition of width t.

Let $\Omega = (v_1, \dots, v_n)$ be a vertex ordering satisfying scalar dimension t. We induct on $|V(G)|$. The base case (n=1) is satisfied immediately. Next, we consider the inductive step. Let G' be the graph obtained from G by taking $V(G') = V(G)$ and $E(G') = \bigcup_{i \in \{0,\dots,n\}} E(G_i^{\Omega})$. By assumption, we can find a tree decomposition $(T, \mathcal{B})$ of $G' - v_1$ corresponding to vertex ordering $(v_2, \dots, v_n)$ (that is, the tree decomposition has width $\max_{i \in \{2,\dots,n\}} \deg_{G_{i-1}^{\Omega}}(v_i)$). From Lemma 3, we can find $x \in V(T)$ such that $N(v_1) \subseteq \mathcal{B}(x)$. Let T' be obtained by adding a leaf y to T adjacent to x; define $\mathcal{B}'(y) = N(v_1) \cup \{v_1\}$ and $\mathcal{B}'(z) := \mathcal{B}(z)$ for all $z \in V(T)$. We can verify that $(T', \mathcal{B}')$ is a tree decomposition of G with treewidth t.

Next, we suppose that G has treewidth t. Let $(T, \mathcal{B})$ be a minimal tree decomposition with width t. Let t_1 be a leaf of T. Label the remaining vertices of T by $t_2, \dots, t_m$ such that t_i is a leaf of $T_i := T - \{t_1, \dots, t_{i-1}\}$. Let s_i be the neighbor of t_i in T_i for $i \in [m-1]$. Let $U_1 := \mathcal{B}(t_1) \setminus \mathcal{B}(s_1)$ which is non-empty by the minimality assumption. For $i \geq 2$, we take $U_i := \mathcal{B}(t_i) \setminus \left(\bigcup_{j=1}^{i-1} U_j \cup \mathcal{B}(s_i) \right)$ (take $U_m := \varnothing$). Let Ω be a vertex ordering of G satisfying $(U_1, U_2, \dots, U_m)$ so that all vertices in U_i are ordered such that they are immediately after vertices in U_{i-1}. By construction, the only neighbors in $G - \bigcup_{j=1}^{i-1} U_j$ each vertex $v \in U_i \subseteq V(G)$ has are in $\mathcal{B}(t_i)$. In particular, this means the scalar dimension of Ω is $\max |\mathcal{B}(t_i)| - 1 = t$.

□

An *edge contraction* of an edge $e = xy \in E(G)$ of a graph is the graph operation resulting in a graph G' by deleting e and identifying the vertices $x = y$ (replace x and y with a vertex whose neighborhood is the union of x and y). We say H is a *minor* of G if it can be obtained from G by a sequence of edge deletions, vertex deletions, and edge contractions.

Since its re-introduction by Robertson and Seymour, treewidth has found applications in relation to graph minors. In particular, the Robertson-Seymour Grid Minor Theorem [8] characterizes graphs with large treewidth as having a large grid graph as a minor. The same theorem was used in Robertson and Seymour's proof of the Graph Minor Theorem [7] – which says that every graph class closed under taking minors can be characterized by forbidding a finite number of graphs as minors.

Another parameter related to treewidth is the treedepth of a graph. The *treedepth* of a graph G is obtained by constructing a set of rooted trees on $V(G)$ in a certain way, considering the length of the longest leaf-to-root in each tree, then letting the treedepth be the shortest such length among these rooted trees. We outline below how each rooted tree is constructed, iteratively:

- Pick some vertex $v^0 \in V(G)$. This will be the root of the tree.
- Consider the components of $G_1 := G - v^0$. Let I_1 be the collection of components of G_1. Pick a vertex v_i^1 from each component of G_1, with $i \in [|I_1|]$. Make each v_i^1 adjacent to v^0 in the tree.

- For $j \geq 2$, let $G_j := G - v_0 - \bigcup_{l=1}^{j-1} \left(\{v_i^l\}_{i\in[|I_l|]}\right)$. Let I_j be the collection of components of G_j. Pick a vertex v_i^j, with $i \in [|I_j|]$ from each component C_i of G_j. For each v_i^j there is a unique $u \in \{v_i^{j-1}\}_{i\in[|I_{j-1}|]}$ that is in the same component of G_{j-1} as v_i^j; make v_i^j adjacent to u.

- When G_j is the empty graph, the resulting graph is the desired tree.

From [6, Inequality (2)], the treedepth of a graph G is related to its longest path (**lp**) as follows:

$$\log_2(\mathbf{lp}(G) + 1) \leq \mathbf{td}(G) \leq \mathbf{lp}(G).$$

From its definition, we can see that $\mathbf{overlap}(G) \leq \mathbf{lp}(G)$, from which it follows that

Observation 5.

$$\mathbf{overlap}(G) \leq 2^{\mathbf{td}(G)} - 1.$$

3 Treewidth and Overlap

In this section, we discuss a weakening of the overlap parameter and some results for graphs satisfying this weaker parameter.

We define **weak-overlap**(G) as follows:

$$\textbf{weak-overlap}(G) := \min_T\{\max\{|S| : S \subseteq E(G) \setminus E(T) \text{ and } \bigcap_{e\in S} E(P_T{}^e) \neq \varnothing\}\}$$

where the minimum is over all spanning trees T of G. From the previous definition, it follows that **weak-overlap**$(G) \geq$ **overlap**(G). The proof for the next lemma is adapted from [2, Lemma 4.3] and shows one way in which overlap relates to treewidth.

Lemma 6. [2] *Fix some $k \in \mathbb{N}$. If G is a graph with max degree Δ and* **weak-overlap**$(G) \geq 2\Delta k$*, then any tree decomposition $(T, \mathcal{B})$ of G such that T is a spanning tree of G and $v \in \mathcal{B}^{-1}(v)$ for all $v \in V(G)$ has* $\text{width}(T, \mathcal{B}) \geq k$.

Proof. Let k, Δ, G, and $(T, \mathcal{B})$ be as described in the lemma statement. We begin by finding an edge-set $S \subseteq E(G) \setminus E(T)$ of size $2\Delta k$ satisfying the **weak-overlap**$(G) \geq 2\Delta k$ condition.

We can deduce that there is a matching of size $k + 1$ inside S. Otherwise, the largest matching of S would be size $\leq k$; there would be $\leq 2(\Delta - 1)k$ edges of S not in the matching and, consequently, $|S| \leq k(1 + 2(\Delta - 1)) < 2\Delta k$ which is a contradiction!

Let $S' := \{u_1v_1, \ldots, u_{k+1}v_{k+1}\}$ be a matching of size $k+1$ in S. Let $x \in \bigcap_{e\in S'} V\left(C_T{}^e\right)$. By construction, x is in the path P^i between u_i and v_i in T for every $i \in \{1, \ldots, k+1\}$. We know that the trees $T_{u_i} := T[\mathcal{B}^{-1}(u_i)]$ and $T_{v_i} := T[\mathcal{B}^{-1}(v_i)]$ intersect because $u_iv_i \in E(G)$. Furthermore, since T_{u_i} and T_{v_i} are connected in T, with $u_i \in T_{u_i}$ and $v_i \in T_{v_i}$, we find that every vertex of $P_{u_iv_i}$ is in $V(T_{u_i}) \cup V(T_{v_i})$. As a result, $x \in V(T_{u_i}) \cup V(T_{v_i})$. That is, $u_i \in \mathcal{B}(x)$ or $v_i \in \mathcal{B}(x)$ for all $i \in \{1, \ldots, k+1\}$. Since S' is a matching, we have that $|\mathcal{B}(x)| \geq k+1$ and the width of $(T, \mathcal{B})$ is at least k. □

Corollary 7. *Let $k \geq 2$ be a positive integer. There is a connected graph G with* $\mathbf{tw}(G) = 2$ *for which any tree decomposition $(T, \mathcal{B})$ of G such that T is a spanning tree of G with $v \in \mathcal{B}^{-1}(v)$ for all $v \in V(G)$ has* $\text{width}(T, \mathcal{B}) \geq k$

Proof. Apply Lemma 6 to the graph obtained from Theorem 1. □

Corollary 7 was used to prove the following result regarding tree decompositions where the trees are minors [2, Theorem 2]:

Theorem 2. [2] *For every positive integer k, there is a connected graph G of treewidth 2 such that if $(T, \mathcal{B})$ is a tree decomposition of G and T is a minor of G, then $(T, \mathcal{B})$ has width at least k.*

Robert Hickingbotham independently proved Corollary 7 in [5, Theorem 7.5.1] using a different approach, but the connection to overlap is new.

4 Preliminaries

In this section, we introduce the graph class of "reflected-trees" which we will use to prove Theorem 1 and prove some basic facts about them.

We say a rooted binary tree has *depth k* if every vertex that is not the root or a leaf has degree 3 and every leaf has the same path length (k) to the root. A *reflected-tree* is a graph G which is obtained by taking two copies of a rooted binary tree with depth k and identifying, or "pasting", the leaves of each copy according to the trivial isomorphism between the copies; furthermore, we say that the binary tree *constructs G*. We write R^k for the reflected-tree constructed from pasting two binary trees of depth $k-1 \geq 1$. We call a

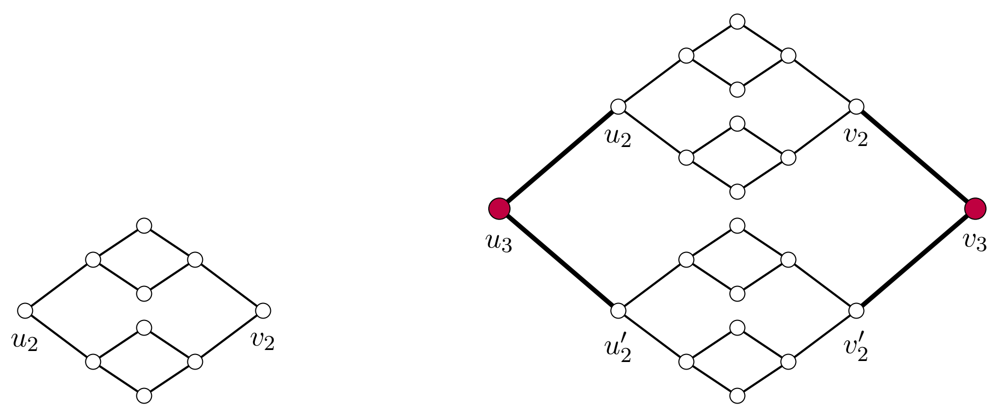

Figure 1: The reflected-tree R^4 of depth 3 (right, with root vertices larger and in red) being constructed from the reflected-tree R^3 of depth 2 (left).

vertex of a reflected-tree G a *root vertex* if it is a root of the rooted binary tree which constructs G. Given a reflected-tree G and a spanning tree T of G, we say that $e \in E(G) \setminus E(T)$ is a *root edge from T* if it is adjacent to a root vertex of G. Whenever the spanning tree is clear, we simply say root edge.

Note that a reflected-tree G with a spanning tree T has at most one root edge from T. If there is a root edge, we say T is a *Type-1* spanning tree; otherwise, we say T is *Type-2*.

If we remove the root vertices of a reflected-tree graph R^{k+1}, the resulting components are isomorphic to R^k (as proven in the following lemma) and we call these *R^k-subgraphs* of R^{k+1} or *reflected-tree subgraphs*. We say a spanning tree T of R^k is *Type-1 everywhere* if, for any R^j-subgraph H of R^k (with $j \leq k$), we have that $T[V(H)]$ is a Type-1 spanning tree of H.

Lemma 8. *If u, v are the root vertices of $G = R^{k+1}$ with $k \geq 2$, then the two components of $G - \{u, v\}$ are isomorphic to R^k.*

Proof. If T is a rooted binary tree of depth $k+1$ and we remove its root, the remaining (two) components are binary trees of depth k. Therefore, from the definition of root vertex and the construction of reflected-trees, it follows that the (two) components of $G - \{u, v\}$ are isomorphic to R^k. □

Corollary 9. *The two R^k-subgraphs of R^{k+1} are disjoint.*

Lemma 10. *Let $k \geq 3$, $G = R^k$, and T be a Type-2 spanning tree of G, $X \subset V(G)$ such that $G[X]$ is an R^{k-1}-subgraph and $T[X]$ is connected. There is an edge $e \in E(G) \setminus (E(G[X]) \cup E(T))$ such that $E(P_T{}^e) \cap E(T[X]) \neq \varnothing$.*

Proof. Let $X' \subseteq V(G)$ be the other vertex-set $X \neq X'$ such that $G[X']$ is an R^{k-1}-subgraph. Note that $T[X']$ is not connected because T is Type-2 (it has no root edges). As a result, we can find some edge $e \in G[X'] \setminus T[X'] \subseteq E(G) \setminus (E(G[X]) \cup E(T))$ such that $T[X'] + e$ is connected. We can deduce that G is the minimal reflected-tree subgraph which contains $C_T{}^e$. In particular, the root vertices of G are in $C_T{}^e$; the path between the root vertices in T intersects $T[X]$ so that $E(P_T{}^e) \cap E(T[X]) \neq \varnothing$. □

We call an edge a *parallel edge* if it is not a unique edge between two vertices.

Definition. A graph is called *Series-Parallel* if it can be constructed from the empty graph using the following operations:

- Adding a vertex of degree ≤ 1.
- Adding a loop or a parallel edge.
- Subdividing an edge.

Note that the definition above allows for non-simple graphs.

Observation 11. *Reflected-trees are Series-Parallel. We can construct R^3 from R^2 using Series-Parallel operations (illustrated in Figure 2). Thus, we can induct on R^k from R^{k-1} using Series-Parallel operations by turning each R^2-subgraph into an R^3.*

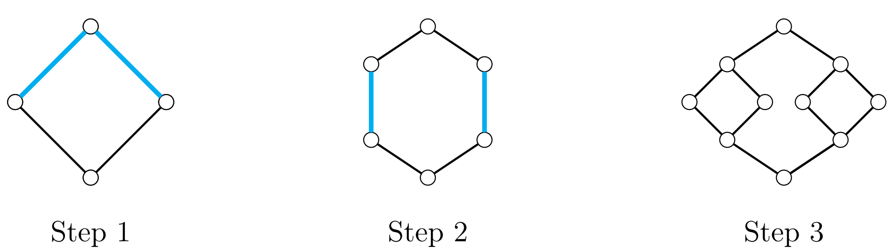

Figure 2: Construction of R^3 from R^2 using Series-Parallel operations. Obtain Step 2 from subdividing the thick blue edges in Step 1. Obtain Step 3 by adding one edge parallel to each thick blue edge in Step 2, then subdividing all four parallel edges.

Lemma 12. *Any reflected-tree G has* $\mathbf{tw}(G) = 2$*.*

Proof. Reflected-trees are connected, but not trees. So, $\mathbf{tw}(G) \geq 2$. To show $\mathbf{tw}(G) \leq 2$ we will show that $\mathbf{tw}(H) \leq 2$ for any series-parallel graph H (reflected-trees are series-parallel). It suffices to show that if H' is obtained from H (with $\mathbf{tw}(H) \leq 2$) from a single Series-Parallel operation, then $\mathbf{tw}(H') \leq 2$. Let $(T, \mathcal{B})$ be a tree decomposition of H with width ≤ 2.

Adding a vertex w of degree ≤ 1: Let $N(w)$ be the set of neighbors of w in H. Pick $t \in V(T)$ such that $N(w) \subseteq \mathcal{B}(s)$ (such t exists because $|N(w)| \leq 1$). Construct a tree T' by adding a leaf t' to T which is only adjacent to t. Construct $\mathcal{B}'$ by extending the map $\mathcal{B}$ such that $\mathcal{B}'(t') = \{w\} \cup N(w)$. Then, $(T', \mathcal{B}')$ is a tree decomposition of H' with width ≤ 2.

Adding a parallel-edge or a loop: Any tree decomposition of H is also a tree decomposition of H' in this case.

Subdividing an edge uv by adding a vertex w: Let $t \in V(T)$ be such that $\mathcal{B}(t) = \{u, v\}$. Construct T' by adding a leaf t' adjacent to t. Extend $\mathcal{B}$ to $\mathcal{B}'$ by defining $\mathcal{B}'(t') = \{u, v, w\}$. The resulting pair $(T', \mathcal{B}')$ is a tree decomposition of H' with width ≤ 2 □

5 Main Result

This section contains the proof of Theorem 1 and key facts used to prove it.

Definition. If G is a reflected-tree and T is a spanning tree of G, we say an ordered set of edges $e_1, \ldots, e_m \in E(G) \setminus E(T)$ has *strictly nested trees* if there are vertex sets $X_1, \ldots, X_m \subseteq V(G)$ satisfying:

- $T[X_1] \subsetneq \ldots \subsetneq T[X_m]$ and, for each $i \in [m]$, $T[X_i]$ is a tree.
- For all $i \in [m]$, X_i is minimal such that $G[X_i]$ is a reflected-tree subgraph of G with $E({C_T}^{e_i}) - e_i \subseteq T[X_i]$.

We say the vertex-sets $X_1, \ldots, X_m$ are *accompanying* if they satisfy the above properties according to $e_1, \ldots, e_m$. This next lemma tells us how edges with strictly nested trees intersect.

Lemma 13. *Let G be a reflected-tree and T a spanning tree of G. If $e_1, e_2 \in E(G) \setminus E(T)$ have strictly nested trees with accompanying vertex-sets X_1, X_2 (respectively) and*

$$E({C_T}^{e_1}) \cap E({C_T}^{e_2}) \neq \varnothing,$$

then

$$E({C_T}^{e_1}) \cap E({C_T}^{e_2}) = E({P_T}^1)$$

where ${P_T}^1$ is the path in T between the root vertices of $G[X_1]$.

Proof. Let u_1, v_1 be the root vertices of $G[X_1]$.

Claim 1.

$$u_1, v_1 \in V({C_T}^{e_1})$$

Proof. Observe that, by minimality of X_1, we have $u_1 \in V({C_T}^{e_1})$ or $v_1 \in V({C_T}^{e_1})$. Assume $u_1 \in V({C_T}^{e_1})$. Since $G[X_1] - \{u_1, v_1\}$ has two components and $u_1 \in V({C_T}^{e_1})$, we have $v_1 \in V({C_T}^{e_1})$ or else ${C_T}^{e_1}$ cannot be a cycle in G. □

A corollary of the previous claim is the following:

$$E({P_T}^1) \subseteq E({C_T}^{e_1}). \tag{1}$$

Since u_1 and v_1 are root vertices of $G[X_1]$, they are also the only vertices in X_1 incident to edges with ends in vertex-set $X_2 \setminus X_1$. In particular, if a path intersects $T[X_1]$ edge-wise and has edges not in $E(T[X_1])$, then it contains the vertices u_1, v_1 and, thus, the path ${P_T}^1$. The path ${C_T}^{e_2} - e_2$ satisfies these properties, so:

$$E({C_T}^{e_1}) \cap E({C_T}^{e_2}) \subseteq E({C_T}^{e_1}) \cap E(T[X_1]) = E({P_T}^1). \tag{2}$$

Using $E({P_T}^1) \subseteq E({C_T}^{e_2})$ along with (1) and (2), we can obtain our result that:

$$E({C_T}^{e_1}) \cap E({C_T}^{e_2}) = E({P_T}^1).$$

□

Lemma 14. *Let G be a reflected-tree of depth $k \geq 1$ and T be a spanning tree of G. For all $m \leq k$, there is an ordered set of edges $e_1, \ldots, e_m \in E(G) \setminus E(T)$ that has strictly nested trees and satisfying the following (if $m \geq 2$)*

$$\bigcap_{i=1}^{m} E({C_T}^{e_i}) = E({P_T}^1)$$

where ${P_T}^1$ is the path in T between the root vertices of $G[X_1]$ and X_1 is the vertex-set accompanying e_1.

Proof. We begin by constructing a set of edges with strictly nested trees. It suffices to prove the claim for $m = k$ (a subset of an edge-set with strictly nested trees also has strictly nested trees). We can induct on the depth k. For the base case, $k = 1$ and G is a cycle on four vertices; $E(G) \setminus E(T)$ is a single edge and it satisfies the strictly nested trees property.

For the inductive step, assume the claim holds for any reflected-tree of depth $k-1$ (and spanning tree of it) and let G be a reflected-tree of depth k. Let Y and Y' be the vertex sets for the two R^k-subgraphs of G. We may assume $T[Y]$ is the component which intersects the path in T between the root vertices of G edge-wise. From this choice, $T[Y]$ is connected. By assumption, we can find a set of edges $e_1, \ldots, e_{k-1} \in E(G[Y]) \setminus E(T[Y])$ that has strictly nested trees corresponding to $X_1, \ldots, X_{k-1}$; note that $T[X_{k-1}] \subseteq T[Y]$. Let $X_k := V(G)$. If T is Type-1, we let e_k be the root-edge from T; otherwise, $T[Y']$ is not connected, and pick e_k according to Lemma 10. In both cases, we see that $E({C_T}^{e_k}) - e_k \nsubseteq T[Y]$ and $E({C_T}^{e_k}) - e_k \nsubseteq T[Y']$. Hence, X_k is minimal such that $T[X_k] \supsetneq T[X_{k-1}]$ is a tree and $G[X_k]$ is a reflected-tree subgraph of G with $E({C_T}^{e_k}) - e_k \nsubseteq T[X_k]$. As a result, if G has depth k, the edge-set $e_1, \ldots, e_k$ has strictly nested trees and is size k.

Next, we prove the second part of the lemma. By our inductive construction of $e_1, \ldots, e_m$ we can apply Lemma 13 for $m > 2$ repeatedly and the desired result follows:

$$\bigcap_{i=1}^{m} E({C_T}^{e_i}) = E({P_T}^1).$$

□

The lemma that follows tells us that, in some sense, Type-1-everywhere spanning trees minimize the intersection of fundamental cycles of a set of edges of fixed size.

Lemma 15. *Let T be a Type-1-everywhere spanning tree of G, a reflected-tree of depth $k \geq 1$, and T' be a different spanning tree of G. If $E_1 \subseteq E(G) \setminus E(T)$, then there is some $E_2 \subseteq E(G) \setminus E(T')$ with $|E_1| = |E_2|$ and such that*

$$\left| \bigcap_{e \in E_1} E({C_T}^e) \right| \leq \left| \bigcap_{e \in E_2} E({C_{T'}}^e) \right|.$$

Proof. To start the proof, we first prove some facts about E_1.

Claim 2. *We may assume the edges of E_1 can be ordered so that they have strictly nested trees.*

Proof. We begin by noting that for any pair of edges $e, e' \in E_1$ we can find vertex sets $X, X' \subseteq V(G)$ such that $e \in E(G[X])$ and $e \in E(G[X'])$ such that $T[X]$ and $T[X']$ are spanning trees of a minimum-size reflected-tree subgraph. Furthermore, if $T[X]$ and $T[X']$ is disjoint for any such pair of edges, their fundamental cycles (contained in the respective spanning tree) are also disjoint. Consequently, we have $T[X] \subsetneq T[X']$ or $T[X'] \subsetneq T[X]$ because T is Type-1-everywhere.

If $\bigcap_{e \in E_1} E({C_T}^e) = \varnothing$, then the lemma is satisfied vacuously so we assume otherwise and order the edges $e_1, \ldots, e_l \in E_1$ so that they have strictly nested trees with accompanying sets $X_1, \ldots, X_l$. We note that all $T[X_i]$ are distinct – every e_i is a root edge from $T[X_i]$ because T is Type-1-everywhere; as a result, by the minimum-size condition, we have that e_i is a root edge from $T[X_i]$ and each $T[X_i]$ has exactly one root edge. For the same reason and because T is Type-1-everywhere, we can also assume that $|E_1| = l \leq k$. ☐

Lemma 15 holds whenever $l = 1$ because we can take $e'_1 \in E_2$ to either be a root edge of T' (if T' is Type-1) or an edge in the R^k-subgraph of $G = R^{k+1}$ disconnected (in T') which has fundamental cycle of the same length. It only remains to show the lemma holds for $l \geq 2$.

Claim 3. *Lemma 15 holds when $l \geq 2$.*

Proof. The ordered edges $e_1, \ldots, e_l$ of E_1 are strictly nested so they satisfy Lemma 13. The spanning tree T is Type-1-Everywhere so $e_1, \ldots, e_l$ are root edges of some reflected-tree subgraphs and the maximum length of the path between root vertices of $G[X_1]$ is $2(k - l + 1)$. Then, we have

$$\left| \bigcap_{e \in E_1} E({C_T}^e) \right| \leq 2(k - l + 1). \tag{3}$$

Let $f_1, \ldots, f_k$ be the ordered set of edges obtained from Lemma 14 with respect to T' with accompanying vertex-sets $X'_1, \ldots, X'_k$. In the construction of the edge-set from Lemma 14, we found an edge f_i corresponding to a reflected-tree subgraph $G[X_i]$ of depth i for all $i \in [k]$. Relabel $f_{k-l+1}, \ldots, f_k$ as $\{e'_1, \ldots, e'_l\} =: E_2$ (such that E_2 still has strictly nested trees in this ordering). Apply Lemma 13 to E_2 and observe that $G[X'_{k-l+1}]$ is a reflected-tree subgraph of depth $k - l + 1$:

$$2(k - l + 1) = \left| \bigcap_{e \in E_2} E({C_{T'}}^e) \right|. \tag{4}$$

Combining (3) and (4) prove the claim whenever $l \geq 2$. ☐

☐

Lemma 16. *Let $k \geq 2$ be an integer. There is a Type-1-everywhere spanning tree T of R^k for which*

$$\textbf{overlap}(T) = \textbf{overlap}(R^k).$$

Proof. Let T' be a spanning tree T of R^k satisfying $\textbf{overlap}(T') = \textbf{overlap}(R^k)$. If T' is Type-1 everywhere we are done if we set $T = T'$. Otherwise, let T be a spanning tree of R^k. We can apply Lemma 15 to T and T', to deduce that $\textbf{overlap}(T) \leq \textbf{overlap}(T')$ so that $\textbf{overlap}(T) = \textbf{overlap}(R^k)$. □

Theorem 1. *For every integer $k \geq 2$, there is a connected graph G with maximum degree 3 such that* $\textbf{tw}(G) = 2$ *and* $\textbf{overlap}(G) \geq k$.

Proof. Let $j = 3k$. Consider $G = R^j$. By Lemma 16, there is a Type-1-everywhere spanning tree T of G with $\textbf{overlap}(T) = \textbf{overlap}(R^j)$. Let u_j and v_j be the root vertices of G and P^j the path between them in T. From the construction of reflected-trees, we can label the vertices along $P^j = u_j u_{j-1} \dots u_2 u_1 v_2 \dots v_{j-1} v_j$ such that each u_i and v_i are both root vertices of the same R^i-subgraph H_i of G for all $i \in \{1, \dots, j\}$; in this labelling, we let $v_1 = u_1$.

Let $P^i \subseteq P^j$ be the path between u_i and v_i in T. The tree T is Type-1-everywhere so for all $i \geq 2$, we can find a root edge e_i with exactly one of u_i or v_i as an end. In particular, for $l \geq 2$, we have $e_l, \dots, e_j \in E(G) \setminus E(T)$ such that $\bigcap_{i=l}^{j} E(C_T{}^{e_i}) = \bigcap_{i=l}^{j} E(P^i) = E(P^l)$ from the construction of T. Next, consider the edges $e_{\frac{j}{3}+1}, \dots, e_j$, which are k distinct edges. Then,

$$\left| \bigcap_{i=\frac{j}{3}+1}^{j} E(C_T{}^{e_i}) \right| = \left| E(P^{k+1}) \right| = 2k > k.$$

Therefore, $\textbf{overlap}(G) \geq k$ and we are finished. □

6 Future Directions

In this section, we discuss future directions in which this work can be continued. One is in investigating the ways in which pasting two graphs or trees affects the overlap.

Recall that in the construction of reflected-trees, we paste two copies of a binary tree along each other's leaves (by identifying them with their counterpart). We can also consider how pasting these two binary trees along each other's leaves in different ways affects the overlap of its spanning trees. As far as we know, this always result in large overlap and we conjecture the following:

Conjecture 17. *Let H and H' be copies of a rooted binary tree of depth k with vertex-sets of leaves L and L', respectively. For all $k \geq 2$, let $\pi_k : L \to L'$ be a bijective map and:*

- *G_{π_k} be the graph obtained by taking $G = H \cup H'$ but identifying $u = \pi_k(u)$ for all $u \in L$.*

then, for every $c \in \mathbb{N}$ there exists some t such that $\textbf{overlap}(G_{\pi_t}) \geq c$.

There is also a construction where we paste three binary trees instead of two. We conjecture that this graph class also has unbounded overlap:

Conjecture 18. *If $\mathcal{F}$ is the class of graphs where each graph is obtained by pasting three copies of a rooted binary tree of depth $k \in \mathbb{N}$ along their leaves (according to the trivial isomorphism), then for all $c \in \mathbb{N}$ there is some $G \in \mathcal{F}$ satisfying:*

$$\mathbf{overlap}(G) \geq c.$$

More generally, it may be beneficial to investigate how the overlap behaves under pasting two graphs along their leaves. Next, we discuss some graphs with large overlap which we think could be responsible for large overlap in general graphs.

The $(k, p_1, \ldots, p_k)$-*banana* graph is the graph obtained from two vertices that have k (internally) vertex-disjoint paths between them of length $p_1, \ldots, p_k$. More generally, we call these *banana graphs*; whenever $k = p_1 = \ldots = p_k$, we simply call that graph a k-*banana*. Banana graphs are clearly planar and a permutation of $p_1, \ldots, p_k$ does not change the type of banana the graph is. However, we will specify an order of $p_1, \ldots, p_k$ to indicate which embedding into the plane we are using – we let p_1 indicate the length of the top path and p_k the bottom path's length. See Figure 3 for an illustration.

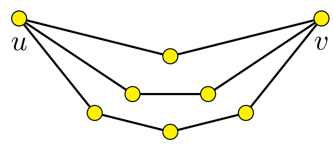

Figure 3: A $(3, 2, 3, 4)$-banana graph with path ends u, v and its indicated embedding into the plane.

We say a graph is a $(k, p_1, \ldots, p_k)$-cycle if it is obtained from a cycle on k vertices by replacing the edges $p_1, \ldots, p_k$ many parallel edges in sequence. We call a graph of this type a *multi-cycle*. Whenever $k = p_1 = \ldots = p_k$, we call the graph a k-multi-cycle. See Figure 4 (right graph) for an illustration.

The *dual* of the planar-embedding of a graph G is obtained by adding a vertex at every face and then adding an edge between two face-vertices for every edge the two corresponding faces share in G. We call the original graph, the *primal* graph. Given a $(k, p_1, \ldots, p_k)$-banana graph it is easy to verify that its dual is a $(k, p_1, \ldots, p_k)$-cycle; that is, a cycle on k vertices with p_i many parallel edges in sequence (see Figure 4).

Unfortunately, bounding overlap by a constant is not closed under taking minors; the next observation shows that it does not always hold that the subgraphs (and thus minors) of a graph G have smaller overlap.

Observation 19. *For each graph H with* $\mathbf{overlap}(H) \geq 3$*, there is a graph G which contains H as a subgraph and has*

$$\mathbf{overlap}(G) < \mathbf{overlap}(H)$$

Proof. Construct G by adding an additional vertex v that is adjacent to all vertices in H. Then, the edges with v as an end form the edge set of a spanning tree $E(T)$ of G. Furthermore, $E(H) = E(G) \setminus E(T)$. The path length between any vertices of G in T is at most 2 so that $\mathbf{overlap}(G) \leq 2 < \mathbf{overlap}(H)$. $\square$

The following two lemmas explain why we might consider k-bananas and k-multi-cycles to be the reason why graphs have large overlap.

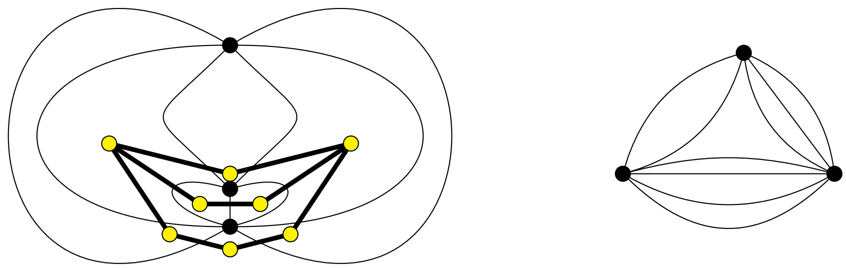

Figure 4: To the left, the dual graph (black vertices with thin edges) of a $(3, 2, 3, 4)$-banana graph (yellow vertices with thick edges). Compare to Figure 3. To the right, the dual graph, a $(3, 2, 3, 4)$-cycle; drawn separately.

Lemma 20. *Let $k \geq 1$ be an integer. If G is a $(k + 1)$-banana, then*

$$\mathbf{overlap}(G) \geq k.$$

Proof. Let T be a spanning tree of G. Let $P_1, \ldots, P_{k+1}$ be the $k + 1$ vertex-disjoint (except their ends) paths of H with ends at $u, v \in G$. Since T is a tree, we can find $e_1, \ldots, e_k \subseteq E(G) \setminus E(T)$ distinct edges so that there is no j for which $e_{i_1}, e_{i_2} \in E(P_j)$ for $j \in [k + 1]$ and $i_1, i_2 \in [k]$. There is also some path $P_i \subset T$. Furthermore, $E(P_i) \subseteq \bigcap_{e \in \{e_1,\ldots,e_k\}} E({P_T}^e)$ so that $\mathbf{overlap}(T) \geq k$. In particular, this means that $\mathbf{overlap}(G) \geq k$ and the proof is finished. □

Lemma 21. *Let $k \geq 1$ be an integer. If G is a $(k + 1)$-multi-cycle, then*

$$\mathbf{overlap}(G) \geq k.$$

Proof. Let T be a spanning tree of G and $V(G) := \{v_1, \ldots, v_{k+1}\}$. Without loss of generality, there must be some $i \in [k]$ for which there are distinct edges $e_1, \ldots, e_k, e_{k+1} \subseteq E(G) \setminus E(T)$ between v_i and v_{i+1}. Since $e_1, \ldots, e_k$ are parallel edges, ${P_T}^{e_1} = \ldots = {P_T}^{e_k}$; furthermore, there are no edges between v_i and v_{i+1} so that the path ${P_T}^{e_1}$ in T is length k. Thus, $\mathbf{overlap}(T) \geq k$ for every spanning tree T of G and $\mathbf{overlap}(G) \geq k$. □

The following two results tell us that if we forbid a planar graph H to construct a class of graphs with small overlap, then we must also forbid its dual. Edges in the primal graph are bijective to edges in the dual graph (in particular, we may associate the primal edges to the dual edges they intersect); so, we denote the dual edge of a primal edge e by e^*. We call a minimal non-empty edge-cut of a connected graph a *bond* and then consider the following proposition from [3, Proposition 4.6.1]:

Lemma 22. [3] *For any connected plane multigraph G, an edge set $E \subseteq E(G)$ is the edge set of a cycle in G if and only if $E^* := \{e^* | e \in E\}$ is a bond in G^*.*

Lemma 23. *If G is a planar graph with a fixed drawing on the plane and G^* is its dual, then*

$$\textbf{overlap}(G) = \textbf{overlap}(G^*).$$

Proof. We begin by noting that the dual G^* is its primal graph, G. Therefore, it will be sufficient to prove that $\textbf{overlap}(G) \leq \textbf{overlap}(G^*)$.

Claim 4. *If T is a spanning tree of G and $E := E(G) \setminus E(T)$, then E^* is the edge-set of some spanning tree T^* of G^**

Proof. Firstly, we observe that a spanning tree of a graph can be characterized as a subgraph which spans the graph and contains exactly one edge from each bond (zero edges from a bond would disconnect the graph and two would create a cycle). Note that $E(G) \setminus E(T)$ is a set of edges such that there is exactly one edge e for each cycle of G. By Lemma 22, this means that each bond has exactly one edge $e^* \in E^*$; as a result, E^* is the edge-set of a spanning tree of G^*. □

Claim 5. *Let T, T^* be as in the previous claim, with $e \in E(G) \setminus E(T)$ and $f \in E(T)$. Then, the edge $f \in E({P_T}^e)$ if and only if $e^* \in E({P_{T^*}}^{f^*})$*

Proof. Claim 4 ensures that the statement of this claim is well-defined. It's sufficient to prove one side of the implication because the dual of G^* is its primal G. The edge $f \in E({P_T}^e)$ if and only if e, f share a cycle in G. By Lemma 22, this occurs if and only if e^*, f^* are in the same bond of G^*. If e^*, f^* are in the same bond then they share a cycle in G^* so that $e^* \in E({P_{T^*}}^{f^*})$. □

Let $t = \textbf{overlap}(G)$. Then, we can find edges $e_1, \ldots, e_t \in E(G) \setminus E(T)$ such that there are edges distinct edges $f_1, \ldots, f_t \in E({P_T}^{e_i})$ for all $i \in [t]$. By Claim 5, it follows that ${f_1}^*, \ldots, {f_t}^* \subseteq E(G^*) \setminus E(T^*)$ satisfy that ${e_i}^* \in E({P_{T^*}}^{{f_i}^*}) \subseteq E(T^*)$ for all $i \in [t]$. By definition, this means that $\textbf{overlap}(G^*) \geq t = \textbf{overlap}(G)$ and the proof is complete.

□

If we wish to characterize the class of graphs with at most overlap c, then we must forbid the dual graphs of every planar graph with overlap $\geq c$. We conjecture that if we can bound the size of a banana or multi-cycle minor in a family of graphs, then we can also bound the overlap of graphs in that family:

Conjecture 24. *Let $m \geq 0$ be an integer. If $\mathcal{F}$ is a family of graphs that has no k-banana minor and no k-multi-cycle minor for $k \geq m$, then there is an integer c_m satisfying*

$$\textbf{overlap}(G) \leq c_m$$

for every $G \in \mathcal{F}$.

Conjecture 24 directly implies a version which forbids more general bananas and multi-cycles as minors. Note that a reflected-tree R^{k+1} of depth k, which has unbounded overlap (as a linear function of k), contains a $(2^{\lfloor k/2 \rfloor}, 2 \cdot \lceil \frac{k}{2} \rceil, \ldots, 2 \cdot \lceil \frac{k}{2} \rceil)$-banana as a minor. To see this, recall that a reflected-tree is constructed from two binary trees of depth k; contract all edges at the $\lfloor k/2 \rfloor$ levels which are closest to the "roots" – this gives two vertices of degree $2^{\lfloor k/2 \rfloor}$ which have that many paths between them of length $2 \cdot (k - \lfloor k/2 \rfloor)$.

7 Acknowledgements

The author would like to thank Rose McCarty for her guidance and Maria Chudnovsky for agreeing to supervise this thesis.

References

[1] Umberto Bertelè and Francesco Brioschi, *Nonserial dynamic programming*, New York: Academic Press, 1972.

[2] Pablo Blanco, Linda Cook, Meike Hatzel, Claire Hilaire, Freddie Illingworth, and Rose McCarty, *On tree decompositions whose trees are minors*, 2023.

[3] R. Diestel, *Graph theory: 5th edition*, Springer Graduate Texts in Mathematics, Springer-Verlag, © Reinhard Diestel, 2017.

[4] Zdeněk Dvořák, *Problem 20 from the Barbados graph theory workshop in 2019*, 2019.

[5] Robert Hickingbotham, *Graph minors and tree decompositions*, B.Sc (Honours) thesis, School of Mathematics, Monash University, Melbourne, Australia, 2019.

[6] Ken-ichi Kawarabayashi and Benjamin Rossman, *A polynomial excluded-minor approximation of treedepth*, Journal of the European Mathematical Society **24** (2022), no. 4, 1449–1470 (en).

[7] Neil Robertson and P. D. Seymour, *Graph minors. XX. Wagner's conjecture*, J. Combin. Theory Ser. B **92** (2004), no. 2, 325–357.

[8] Neil Robertson and P. D. Seymour, *Graph minors. V. Excluding a planar graph*, J. Combin. Theory Ser. B **41** (1986), no. 1, 92–114.